Optimal Mode Changes for Highway Transportation Safety

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***Abstract*—The problem of designing a decentralized advance hazard warning system for highway transportation systems entails the development of efficient switching controllers. In this paper, we represent the vehicle dynamics by a finite number of modes, each of which is represented by a low order transfer function and a constant time delay. The problem of highway safety analysis then gets translated into that of the stability analysis of a parameterized hybrid system. We present an analysis framework and some preliminary original results.**

***Index Terms*—highway safety, Lyapunov functions, hybrid systems, stability**

I. Introduction

A. Motivation

The prevalent brake-light dependent slowdown warning system is not very effective in preventing multiple vehicle collisions. Indeed, rear-end and angle collisions annually account for close to 10 million vehicle crashes in the US [13]. A root cause is that, typically, a driver learns of the hazard by observing brake-lights of only the front vehicle. Severity of the collisions can be reduced if the vehicles have the ability to emit, and respond to, a warning signal over a zone whenever it faces or induces a hazard. The US Department of Transportation has announced the intention of equipping over 10% passenger vehicles and 25% trucks with an advanced slowdown warning system by the year 2010. This paper is motivated by the design of a robust decentralized slowdown warning technology that will significantly improve the highway safety *and* incur a low false alarm rate.

B. Proposed Approach

Our hypothesis is that the combined behavior of a vehicle and a driver can be reliably represented by a finite number of operating modes, each of which can be well approximated by a low order transfer function along with a constant time delay. We consider an arbitrarily large section of a highway and represent the vehicle dynamics therein as an array system indexed in space and time. We assume that the vehicle features are comparable and the operating conditions are uniform. As a result, the hybrid system dynamics are spatially invariant. We represent the dynamics by a family of *partial differential equations* (PDE's) and use a spatial Fourier transform to recast the family of PDE's into a family of *ordinary differential equations* (ODE's) parameterized by the frequency parameter. The notion of highway safety is equivalent to that of the stability of this hybrid system.

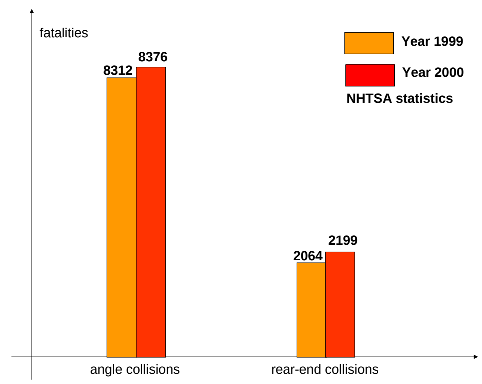

Fig. 1. Fatalities involved in rear-end and angle collisions. Over 10 million such crashes occur every year in the US.

The parameterized state space can be partitioned into cells that share, at most, only each other's boundaries, and the hybrid system has a piecewise affine form in each of the cells. This formulation facilitates the stability analysis using multiple Lyapunov functions, much on the lines of [7]. It turns out that the piecewise quadratic Lyapunov function dependent stability analysis techniques given in [2], [5], and [7] are conservative if applied directly; the reason being that a direct application requires that the Lyapunov functions be independent of the frequency parameter and, consequently, the full benefit of using multiple *local* Lyapunov functions, as opposed to using a single *global* Lyapunov function, is not realized. Essentially, this paper draws on the techniques developed in [1], [4], and [7] to help solve the transportation safety problem on hand.

C. Organization of the Paper

This paper is organized as follows. The notation is introduced in Section II-A. The key relevant concepts are defined in Section II-B. A model for highway safety analysis is described in Section III-A and the safety analysis problem is formulated in Section III-B. The problem solutions are presented in Section V, immediately after the prior art description in Section IV. Comparisons with the existing works are presented in Section VI. The paper is concluded in Section VII.

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II. Preliminaries

A. Notation

The notation is introduced as and when necessary. Capital letter symbols, such as F and G, denote operators whereas small letter symbols, such as x and y, denote real signals which may possibly be vector valued or matrix valued. The set of all real (complex) numbers is denoted $\mathbb{R}$ ($\mathbb{C}$) and the set of all integers is denoted $\mathbb{Z}$. The notation $\doteq$ stands for 'defined as'. The inner product $\langle x, y\rangle \doteq \displaystyle\int_{-\infty}^{\infty} y(t)^T x(t)\ dt$. The norm $\|x\| \doteq \langle x, x\rangle$. The vector space of signals for which the Euclidean norm exists is denoted $\mathcal{L}_2^n$. The vector space $\mathcal{L}_2^n$ is generally referred to as $\mathcal{L}_2$. Fourier transform of x is denoted $\widehat{x}$. Conjugate transpose of a vector or matrix $(\cdot)$ is denoted $(\cdot)^*$; its transpose is denoted $(\cdot)^T$. The (i,j)-th entry of a matrix $(\cdot)$ is denoted as either $(\cdot)_{i,j}$ or $(\cdot)_{ij}$, depending on the ease of reading. Given a two dimensional array $(\cdot)$, the *neighborhood* $\mathcal{N}_k$ of the element $(\cdot)_{i,j}$ denotes the set

$$\{(\cdot)_{mn} \mid |m-i| \le k \text{ and } |n-j| \le k\}.$$

A block diagonal matrix $D \in \mathbb{R}^{n\times n}$ having matrices D_{ii} on its diagonal is denoted $\mathrm{diag}(D_{11}, D_{22}, \ldots, D_{nn})$. Time derivative of the signal x is denoted $\dot{x}$. The class of bounded linear operators mapping the vector space X into the vector space Y is denoted $\mathcal{L}(X,Y)$; the class $\mathcal{L}(X,X)$ is referred to as $\mathcal{L}(X)$. Domain of an operator A is denoted $\mathcal{D}(A)$. The group theoretic and system theoretic notions and concepts that are left undefined may be found in [3] and [8].

B. Definitions

Definition 1: A C_0-semigroup $T(t)$ on a Hilbert space Z is said to be *exponentially stable* if there exist positive constants M and α such that

$$\|T(t)\| \le M e^{-\alpha t} \quad \forall t \ge 0.$$

The supremum over all possible values of α is said to be its *stability margin*. □

Definition 2: [Piecewise Affine Systems, [7]]
The class $\mathcal{S}_H$ of hybrid systems is defined by family of ordinary differential equations as:

$$\dot{x}(t) = A_i x(t) + a_i \quad \forall x(t) \in X_i$$

where $A_i \in \mathbb{R}^{n\times n}, a_i \in \mathbb{R}^n$, and $\{X_i\}_{i\in I} \subset \mathbb{R}^n$ is a partition of the parameterized state-space into a finite number of closed, and possibly unbounded, polyhedral cells with pairwise disjoint interior. The set of cells that include the origin is denoted I_0 and its compliment is denoted I_1. □

Definition 3: [Parameterized Hybrid Systems]
The class $\mathcal{S}_{H\theta}$ of hybrid systems is defined by family of ordinary differential equations, parameterized by the frequency parameter $\theta \in [-\pi, \pi]$, as:

$$\dot{x}_\theta(t) = A_{i,\theta} x(t) + a_{i,\theta} \quad \forall x_\theta(t) \in X_{i\theta}$$

where $A_{i\theta} \in \mathbb{R}^{n\times n}, a_{i\theta} \in \mathbb{R}^n$, and $\{X_{i\theta}\}_{i\in I} \subset \mathbb{R}^n$ is a partition of the parameterized state-space into a finite number of closed, and possibly unbounded, polyhedral cells with pairwise disjoint interior. □

III. Model Description and Problem Formulation

A. Model Description

The state space description of our model is as follows. The state vector x comprises the position, the velocity, and the acceleration of each vehicle. In the *normal* mode of operation, a vehicle adjusts its velocity and acceleration based on the state variables of the vehicles within an $\mathcal{N}_1$-neighborhood. We refer to this mode of operation as normal because in real life, as of now, a vehicle driver gets only so much visual feedback — assuming he is not driving an SUV, so to say. In this mode, the dynamical state space equations describing the k-th vehicle are:

$$\dot{x}_k(t) = A_{0,0}x_k(t) + A_{-1,0}x_{k-1}(t) + A_{1,0}x_{k+1}(t) \tag{1}$$

where $A_{0,0},\ A_{-1,0}$ and $A_{1,0}$ are constant matrices of appropriate dimensions. When all of the vehicles are in the normal mode, the state-space description is given by:

$$\dot{x}(t) = \widetilde{A}_N x(t) \tag{2}$$

where the associated real valued matrix $\widetilde{A}_N$ of the operator $\widetilde{A}_N : \mathcal{L}_2 \to \mathcal{L}_2$ is the band block diagonal matrix

$$\widetilde{A}_N \doteq \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & A_{-1,0} & A_{0,0} & A_{1,0} & \ddots & \ddots & \ddots \\ \ddots & \ddots & A_{-1,0} & A_{0,0} & A_{1,0} & \ddots & \ddots \\ \ddots & \ddots & \ddots & A_{-1,0} & A_{0,0} & A_{1,0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with the entries $(\cdot)_{i,j} = 0\ \forall j$ such that $|j-i| > 6$. When a particular vehicle enters the *cautious* mode, the width of this band increases for the corresponding entries in the system matrix. Specifically, the vehicle adjusts its velocity and acceleration based on the state variables of the vehicles within an $\mathcal{N}_K$-neighborhood. In this mode, the dynamical state space equations describing the k-th vehicle are:

$$\dot{x}_k(t) = A_{0,0}x_k(t) + \sum_{i=1}^{K} A_{-i,0}x_{k-i}(t) + \sum_{i=1}^{K} A_{i,0}x_{k+i}(t) \tag{3}$$

where $A_{0,0},\ A_{-i,0}$ and $A_{i,0}$ are constant matrices of appropriate dimensions. In principle, every element of the set $\{A_{0,0}, A_{-1,0}, A_{1,0}\}$ may be different in the normal mode from its cautious mode value. However, we denote them just the same in order to keep the notation simple. When all of the vehicles are in the cautious mode, the state-space description has the form:

$$\dot{x}(t) = \widetilde{A}_C x(t) \tag{4}$$

where the associated real valued matrix $\widetilde{A}_C$ of the operator $\widetilde{A}_C : \mathcal{L}_2 \to \mathcal{L}_2$ is band block diagonal with the entries $(\cdot)_{i,j} = 0\ \forall j$ such that $|j-i| \ge 6K$. Stability margin of the system increases monotonically with the number of vehicles in the cautious mode and, in particular, stability margin of the system described by (4) is higher than that of the system described by (2). Taking a spatial Fourier transform with the vehicle positions as the frequency parameter, the parameterized hybrid system equations are given as:

$$\dot{x}_\theta(t) = \widetilde{A}_{i\theta} x_\theta(t) \ \ \forall \ x_\theta \in X_{i\theta} \tag{5}$$

where the associated diagonal matrix $\widetilde{A}_{i\theta}$ has entries

$$(\cdot)_{m,m} \doteq A_{0,0} + \sum_{k=1}^{K} A_{-k,0} e^{-jk\theta} + \sum_{k=1}^{K} A_{k,0} e^{jk\theta} \tag{6}$$

on its diagonal where $\theta \in [-\pi/K, \pi/K]$ and the parameterized state space has the partition $\{X_{i\theta}\}$. Note that the system given by (5) and (6) is an instance of the class $\mathcal{S}_{H\theta}$. We denote this system as $\mathcal{S}$. As the vehicles change modes and θ ranges over $[-\pi/K, \pi/K]$, the associated matrix of the system takes values inside a polytope. We denote this polytope as $\mathcal{A}_\Theta$.

B. Problem Formulation

Problem 1: Determine the computationally tractable analytical conditions under which the system $\mathcal{S}$ is stable. □

IV. Prior Art

Lemma 1: [Lemma 5.1.2, [3]]
The C_0-semigroup $T(t)$ on a Hilbert space Z is exponentially stable if and only if there exists a $\gamma_z < \infty$ such that

$$\int_{-\infty}^{\infty} \|T(t)z\|^2 dt \quad \leq \quad \gamma_z$$

for every $z \in Z$. □

Lemma 2: [Theorem 5.1.3, [3]]
Suppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space Z. Then, $T(t)$ is exponentially stable if and only if there exists a positive operator $P \in \mathcal{L}(Z)$ such that

$$\langle Az, Pz \rangle + \langle Pz, Az \rangle < 0$$

for all $z \in \mathcal{D}(A)$. □

We next note down a well known result, viz. [7], on the stability analysis of hybrid systems using multiple Lyapunov functions. Consider $\mathcal{S}_H$. Denote

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}.$$

Construct $\bar{E}_i = [E_i \ e_i], \ \bar{F}_i = [F_i \ f_i]$ where $(e_i, f_i) = (0, 0)$ for all $i \in I_0$ and

$$\bar{E}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \quad \forall \quad x \in X_i, \ i \in I;$$
$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix} \quad \forall \quad x \in X_i \cap X_j, \ i, j \in I.$$

Lemma 3: [Theorem 1, [7]]
Consider symmetric matrices T, U_i, and W_i such that U_i and W_i have non negative entries while

$$P_i \doteq F_i^T T F_i \ \ \forall i \in I_0, \qquad \bar{P}_j \doteq \bar{F}_j^T T \bar{F}_j \ \ \forall j \in I_1$$

satisfy

$$A_i^T P_i + P_i A_i + E_i^T U_i E_i \quad < \quad 0 \tag{7}$$
$$P_i - E_i^T W_i E_i \quad > \quad 0 \tag{8}$$
$$\bar{A}_j^T \bar{P}_j + \bar{P}_j \bar{A}_j + \bar{E}_j^T U_j \bar{E}_j \quad < \quad 0 \tag{9}$$
$$\bar{P}_j - \bar{E}_j^T W_j \bar{E}_j \quad > \quad 0 \tag{10}$$

for all $i \in I_0$ and for all $j \in I_1$. Then, every piecewise continuous trajectory of $\mathcal{S}_H$ tends to zero exponentially. □

V. Main Results

Theorem 1: [Solution to Problem 1]
Consider symmetric matrices T, U_i, and W_i such that U_i and W_i have non negative entries. Suppose $P_i = F_i^T T F_i$ satisfy

$$A_i^T P_i + P_i A_i + E_i^T U_i E_i \quad < \quad 0$$
$$P_i - E_i^T W_i E_i \quad > \quad 0$$

for all A_i that are vertices of $\mathcal{A}_\Theta$ for all $i \in I$. Then, every piecewise continuous trajectory of $\mathcal{S}_H$ tends to zero exponentially. □

Proof: The proof follows on the lines of the proof of Lemma 3. The conditions (7) and (8) need be checked over uncountably infinitely many $\theta \in [-\pi/K, \pi/K]$. However, as θ ranges over $[-\pi/K, \pi/K]$ the matrix A_i takes on values inside the polytope $\mathcal{A}_\Theta$ so that, due to convexity, the conditions need only be checked on the vertices of the polytope $\mathcal{A}_\Theta$. QED.

VI. Discussion

Time delays constitute a critical factor in the highway accidents. In this paper, it is assumed that their effect can be well approximated by finite order rational transfer functions, such as the Padé approximations. A more sophisticated approach to mitigate this pain begins with replacing (1) and (3) by, respectively, the following retarded differential equations:

$$\dot{x}_k(t) \quad = \quad \sum_{i=-1}^{1} A_{i,0} x_k(t - i * h)$$
$$\dot{x}_k(t) \quad = \quad \sum_{i=-K}^{K} A_{-i,0} x_{k-i}(t - i * h)$$

where h denotes the combined reaction time delay of the driver and the vehicle; the value of h can be a function of the operating mode.

Remark 1: The characterization given by Theorem 1 relies on the one given in Lemma 3 which requires that the system be known perfectly. A more comprehensive and computation friendly characterization can be obtained by casting the above algebraic conditions as *integral quadratic constraints* (IQC's) [12]. An IQC based characterization the effect of a localized control on the global stability of a spatially invariant system has been recently derived, see [4, Theorem 1], by [4]. □

Remark 2: Per se, the results established by [1] and [4] address the case of uniformly distributed spatially invariant systems operating in single mode although the bare bone assumptions of [1] can address a much broader class of systems.

[4] methodically builds on the classic framework developed in [1] to characterize the effect of a localized control on the global stability of such systems. A similar characterization for the case of systems operating in multiple modes is crucial for the development of the false alarm mitigation logic referred to in Remark 4. □

Remark 3: String stability of vehicle platoons has been extensively studied as a part of the highway safety initiative in the state of California's PATH program over the last two decades (see, e.g., [9], [14], and [15]). However, the established string stability results concern only continuous time systems sans time delays. The analysis framework used in this paper addresses the realistic case of multiple operating modes which results due to the presence of a human operator in the control loop. In principle, the framework can be used to give less conservative stability conditions than the ones derived for second order macroscopic models by [19]. □

Remark 4: The cautious mode of operation has a higher stability margin than the normal mode of operation. However, since a human driver is assumed to be present in the control loop, it also carries a higher operating cost because cautious behavior incurs fatigue. In order to reduce the number of false alarms, the warning device should have the intelligence embedded in it as to what an optimal control law is when the objective is to minimize a performance function which is expressed in terms of the mode stability margins and operating costs. We have not addressed the synthesis of such a controller. However, a solution would be to assign a high control bandwidth to the system in cautious mode and a low control bandwidth to the system in normal mode. The controller synthesis problem then reduces to synthesizing an LQG/LQR controller on the lines of [6]. □

Remark 5: It is natural to investigate whether the extensive literature on aircraft collision (see, e.g. [16], [17], [18], and references therein) can be used in designing decentralized controllers for vehicle platoon safety. It turns out that the prevalent game theoretic approaches, e.g. [16], are not directly applicable to the problem on hand and the reason is as follows. The hybrid automaton representing a platoon of N vehicles can be described as follows (see [10], [11], and [16] for hybrid automata definitions). The set of discrete states comprises the operating mode of platoon: for example, one particular operating mode corresponds to all vehicles being in cautious mode; this set has 2^{N-1} elements. The set of states that are continuous in time comprises the inter-vehicle separations x_i. The set of discrete inputs and disturbances comprises the hazard signals. The set of continuous in time control inputs comprises accelerations of the platoon vehicles. The reset function sets a vehicle operating mode to cautious if it has received a hazard signal and resets it to normal when it is appropriate. This automata is said to be safe if the inter-vehicle separations are positive at any given time. The optimal controller to guarantee safety can be computed as the solution to a dynamic game between the control and disturbance [16]; the value function of the game over the horizon $[0, T]$ is $J(x, \sigma, u, 0) \doteq \min_i\{x_i(T)\}$ where $i \in \{1, 2, \ldots, N-1\}$. Solution to this problem is obtained by solving an optimal control problem to generate the switching signals σ and the control inputs u, and by obtaining the corresponding worst case disturbance. In our highway safety problem, the worst disturbance corresponds to the case in which none of the vehicles are equipped and the first car slows down to a standstill as fast as it can. A decentralized controller must consider switching patterns of all vehicles and the front car velocity profiles as the disturbance. An implementation of this methodology to compute safe sets for more than two vehicles is not described in [16] and suffers from the curse of dimensionality. Unsafe sets can in fact be defined only for each independently maneuvered vehicle separately in the formulation of [16]. This difficulty can be resolved by taking a union of unsafe sets of each vehicle in order to generate a *working* unsafe set. However, this solution is liable to be very conservative and will not be internally consistent because each vehicle must assume the worst behavior of every other. □

VII. Conclusion

The problem of designing a decentralized advance hazard warning system for highway transportation systems entails the development of efficient switching controllers. In this paper, we have represented the highway dynamics by a finite number of modes, each of which is represented by a low order transfer function and a constant time delay. The problem of highway safety analysis then gets translated into that of the stability analysis of a parameterized hybrid system. We have presented a preliminary stability analysis result based on the well known techniques of [1], [4], and [7]. Connections with the existing literature and ongoing research activities are noted down. ■

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