Agreement Problems in Networks with Directed Graphs and Switching Topology

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Abstract

In this paper, we provide tools for convergence and performance analysis of an agreement protocol for a network of integrator agents with directed information flow. We also analyze algorithmic robustness of this consensus protocol for networks with mobile nodes and switching topology. A connection is established between the Fiedler eigenvalue of the graph Laplacian and the performance of this agreement protocol. We demonstrate that a class of directed graphs, called balanced graphs, have a crucial role in solving average-consensus problems. Based on the properties of balanced graphs, a group disagreement function (i.e. Lyapunov function) is proposed for convergence analysis of this agreement protocol for networks with directed graphs and switching topology.

1 Introduction

Distributed decision-making for coordination of networks of dynamic agents has attracted several researchers in recent years. This is partly due to broad applications of multi-agent systems in many areas including cooperative control of unmanned air vehicles (UAVs), flocking of birds, schooling for underwater vehicles, distributed sensor networks, attitude alignment for cluster of satellites, and congestion control in communication networks.

Agreement problems have a long history in the field of computer science, particularly in automata theory and distributed computation [8]. In many applications involving multi-agent/multi-vehicle systems, groups of agents need to agree upon certain quantities of interest. Such quantities might or might not be related to the motion of the individual agents. As a result, it is important to address agreement problems in their general form for networks of dynamic agents with directed information flow under link failure and creation (i.e. networks with switching topology).

Our main contribution in this paper is to provide convergence, performance, and robustness analysis of an agreement protocol for a network of integrator agents with directed information flow and (perhaps) switching topology.

In the past, a number of researchers have worked in problems that are essentially different forms of agreement problems with differences regarding the types of agent dynamics, the properties of the graphs, and the names of the tasks of interest [1, 2, 7, 9]. The work of Jadbabaie *et al.* in [6] focuses on attitude alignment for network of agents with an undirected graph in which each agent has a discrete-time integrator dynamics. It is shown that the connectivity of union of the graphs in a sufficiently large time interval is sufficient for convergence of the heading angles of the agents. In [11], the authors addressed convergence of linear and nonlinear protocols for networks with undirected graphs in presence or lack of communication time-delays. Consensus problems for directed graphs is rather challenging and has not been systematically considered before.

In this paper, we provide convergence analysis of an agreement protocol for a network of integrators with a directed information flow and fixed or switching topology. Our analysis relies on several tools from algebraic graph theory [4] and matrix theory [5]. We establish a connection between the performance of this consensus protocol and the Fiedler eigenvalue of graph Laplacian. The interpretation of the Fiedler eigenvalue of a digraph was unknown. Here, we introduce the notions of balanced graphs and the mirror of a digraph that allow extension of the notion of the Fiedler eigenvalue (i.e. algebraic connectivity) to digraphs. It is demonstrated that balanced digraphs a crucial role in derivation of Lyapunov functions for convergence analysis of average-consensus problems on directed networks with fixed or switching topology.

An outline of this paper is as follows. In Section 2, we provide some background on algebraic graph theory. In Section 3, we present the setup for agreement problems in directed networks with switching topology. Our main results are given in Section 4. In Section 5, the simulation results are presented. Finally, in Section 6, we make our concluding remarks.

2 Preliminaries: Algebraic Graph Theory

In this section, we introduce some basic concepts and notation in graph theory that will be used throughout the paper. More information is available in [4].

Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted directed graph (or digraph) with n nodes and a *weighted adjacency matrix* $\mathcal{A} = [a_{ij}]$ where $a_{ij} \geq 0$ for all $i, j \in \mathcal{I} = \{1, 2, \dots, n\} : i \neq j$ and $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of *neighbors* of the node v_i is denoted by N_i and defined as $N_i = \{j \in \mathcal{I} : a_{ij} > 0\}$. The in–degree and out–degree of node v_i are, respectively, defined as follows:

$$\deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}, \ \deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}. \tag{1}$$

For an ordinary graph with $\mathcal{A}$ that has binary elements $\deg_{out}(v_i) = |N_i|$. The *degree matrix* of G is a diagonal matrix denoted by $\Delta = [\Delta_{ij}]$ where $\Delta_{ij} = 0$ for all $i \neq j$ and $\Delta_{ii} = \deg_{out}(v_i)$. The (weighted) *graph Laplacian matrix* associated with G is defined as

$$L = \mathcal{L}(G) = \Delta - \mathcal{A}. \tag{2}$$

With a slight misuse of notation, we use $\mathcal{L}(G) = \mathcal{L}(\mathcal{A})$ to denote the Laplacian of graph G. By definition, the graph Laplacian has an eigenvector at $\lambda_1 = 0$ and a right eigenvector $w_r = \mathbf{1} = (1, 1, \dots, 1)^T$ with identical nonzero elements. Furthermore, for a *strongly connected* digraph G of order n, the Laplacian matrix satisfies the following rank condition:

$$\text{rank}(L) = n - 1 \tag{3}$$

A digraph is called strongly connected if and only if any two distinct nodes of the graph can be connected via a path that respects the orientation of the edges of the digraph.

Note. Throughout this paper, we assume all graphs have at least two nodes (i.e. are non-trivial) and there is no cycle of length one (i.e. an edge from a node to itself).

For an *undirected graph* G, L is symmetric and positive semi-definite. The *disagreement function* (also referred to as Laplacian potential) associated with G is defined (up to a positive factor) in [11] as follows

$$\Phi_G(x) = x^T L x = \frac{1}{2} \sum_{ij \in \mathcal{E}} (x_j - x_i)^2 \tag{4}$$

where x_i denotes the value of node v_i. The value of a node might represent physical quantities including attitude, position, temperature, voltage, and so on. We say two distinct nodes v_i and v_j *agree* if and only if $x_i = x_j$. Apparently, $\Phi_G(x) = 0$ if and only if all neighboring nodes in G agree. If in addition, the graph is connected, then all nodes in the graph agree and a *consensus* is reached. Therefore, $\Phi_G(x)$ is a meaningful function that quantifies the group disagreement in a network.

For an undirected graph G that is connected the following well-known property holds [4]:

$$\min_{\substack{x \neq 0 \\ \mathbf{1}^T x = 0}} \frac{x^T L x}{\|x\|^2} = \lambda_2(L) \tag{5}$$

The proof follows from a special case of Courant–Fischer Theorem in [5]. We will later establish a connection between $\lambda_2(\hat{L})$, called the *Fiedler eigenvalue* of $\hat{L}$ [3], and the *performance* of a linear agreement protocol where $\hat{L}$ is closely-related to L.

Later, we use the spectral properties of graph Laplacians for convergence analysis of the main agreement protocol in this paper. The following result follows from Geršgorin disk theorem [5]:

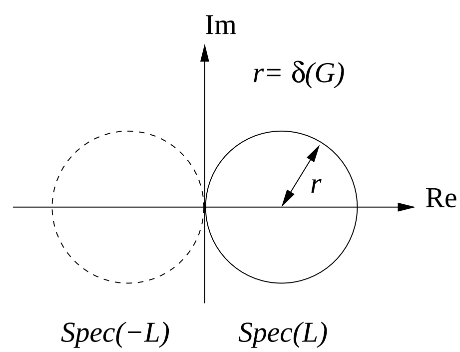


Figure 1: A demonstration of Geršgorin Theorem applied to graph Laplacian.

Proposition 1. *(spectral localization) Let* $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ *be a digraph with the Laplacian* L. *Denote the maximum node out–degree of* G *by* $\delta(G) = \max_i \deg_{out}(v_i)$. *Then, all the eigenvalues of* $L = \mathcal{L}(G)$ *are located in the following disk*

$$D(G) = \{z \in \mathbb{C} : |z - \delta(G)| \leq \delta(G)\} \tag{6}$$

centered at $z = \delta(G) + 0j$ *in the complex plane (see Figure 1). Moreover, the real-part of the eigenvalues of* $-L$ *are non-positive.*

Proof. See [10]. □

3 Agreement Problem on Directed Graphs

Consider a network of integrators

$$\dot{x}_i = u_i, \ i \in \mathcal{I}, x_i, u_i \in \mathbb{R}$$

with *information flow* (or *topology*) $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. On purpose, we choose agents with simple linear dynamics to demonstrate issues involved in the networks and communication aspects of agreement.

The *asymptotic agreement problem* can be described as follows. Give a protocol that guarantees the state of the network as a whole asymptotically converges to an equilibrium state $x^* \in \mathbb{R}^n$ with identical elements, i.e. $x_i^* = x_j^* =: \alpha$ for all $i, j \in \mathcal{I}, i \neq j$. The element α that determines x^* is called the *group decision value*. An agreement problem in which $\alpha = Ave(x(0))$ is referred to as the *average-consensus problem* [11] where $Ave(x) = (\sum_{i=1}^n x_i)/n$.

We focus on solving the *average-consensus problem* using the following *agreement protocol*:

$$u_i(t) = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)),\ i \in \mathcal{I} \tag{A}$$

In *distributed average-consensus* problem, the objective of each node of the network is to calculate the average of the initial values of all n nodes provided that no node has an edge with all other nodes (unless $n = 2$) and the network is connected.

Given Protocol (A), the state of the network evolves according to the following linear system

$$\dot{x}(t) = -Lx(t) \tag{7}$$

where $L = \mathcal{L}(G)$ is the Laplacian induced by the information flow G. In a network with variable topology G, convergence analysis of Protocol (A) is equivalent to stability analysis for a *hybrid system*

$$\dot{x}(t) = -L_k x(t), \quad k = s(t) \tag{8}$$

where $L_k = \mathcal{L}(G_k)$ is the Laplacian of G_k, $s(t) : \mathbb{R} \to \mathcal{I}_\Gamma \subset \mathbb{Z}$ is a switching signal, and $\Gamma \ni G_k$ is a finite collection of digraphs (of order n) with the index set $\mathcal{I}_\Gamma$. The task of stability analysis for the hybrid system in (8) is rather challenging partly because $q, p \in \mathcal{I}_\Gamma, q \neq p$ most likely implies $L_q L_p \neq L_p L_q$. Thus, rather simple ways of constructing a common Lyapunov function fail for this switching system.

The following result guarantees the convergence of Protocol (A) for digraphs.

Proposition 2. *Consider a network of integrators with an information flow G that is a strongly connected digraph. Then, Protocol* (A) *globally asymptotically solves an agreement problem, i.e. the solution asymptotically converges to an equilibrium x^* such that $x_i^* = x_j^*$ for all $i, j, i \neq j$.*

Proof. See [10]. □

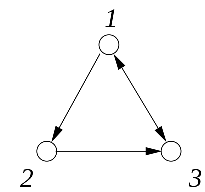


Figure 2: A connected digraph of order 3 that does not solve the average-consensus problem.

Keep in mind that Proposition 2 does not address the average-consensus problem. A sufficient condition for the decision value of each node α to be equal to $Ave(x(0)$ is that $\sum_{i=1}^n u_i \equiv 0$. If G is undirected (i.e. $a_{ij} = a_{ji} > 0, \forall i, j : a_{ij} \neq 0$), automatically the condition $\sum_{i=1}^n u_i = 0, \forall x$ holds and $Ave(x(t))$ is an invariant quantity [11]. However, this property does not hold for a general digraph. A counterexample is given in Figure 2 (see [10] for further details). The existence of digraphs that do not solve average-consensus problems motivates us to *characterize the class of all digraphs that solve the average-consensus problem.*

Before presenting our main results, we need to provide a limit theorem for exponential matrices of the form $\exp(-Lt)$. This is because the solution of (7) is given by

$$x(t) = \exp(-Lt)x(0) \tag{9}$$

and by explicit calculation of $\exp(-Lt)$, one can obtain the decision value of each node after reaching consensus.

Notation. Following the notation in [5], we denote the set of $m \times n$ real matrices by $M_{m,n}$ and the set of square $n \times n$ matrices by M_n. Furthermore, throughout this paper, the right and left eigenvectors of the Laplacian L associated with $\lambda_1 = 0$ are denoted by w_r and w_l, respectively.

Theorem 1. *Assume G is a strongly connected digraph with Laplacian L satisfying $Lw_r = 0$, $w_l^T L = 0$, and $w_l^T w_r = 1$. Then*

$$R = \lim_{t \to +\infty} \exp(-Lt) = w_r w_l^T \in M_n \tag{10}$$

Proof. See [10]. □

4 Main Results

In this section, we present three of our main results: i) characterization of all connected digraphs that solve average-consensus problem using Protocol (A), and ii) the relation between the performance of Protocol (A) and the Fiedler eigenvalue (i.e. algebraic connectivity) of graphs, and iii) robust agreement under switching information flows.

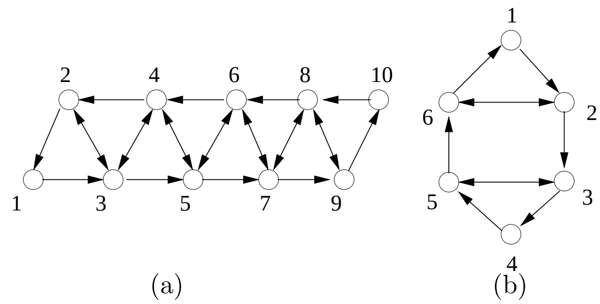


Figure 3: Three examples of balanced graphs.

4.1 Balanced Graphs and Average-Consensus

The following class of digraphs turns out to be instrumental in solving average-consensus problems:

Definition 1. (balanced graphs) We say the node v_i of a digraph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is *balanced* if and only if its in-degree and out-degree are equal, i.e. $\deg_{out}(v_i) = \deg_{in}(v_i)$. A graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is called *balanced* if and only if all of its nodes are balanced, i.e. $\sum_j a_{ij} = \sum_j a_{ji}, \forall i$.

Example 1. *Any undirected graph is balanced.* Furthermore, both of the digraphs shown in Figure 3 are balanced.

Here is our first main result:

Theorem 2. *Consider a network of integrators with directed information flow $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that is strongly connected. Then, G globally asymptotically solves the average-consensus problem using Protocol* (A) *if and only if G is balanced.*

Proof. The proof follows from Propositions 3 and 4. □

Remark 1. According to Theorem 2, if a graph is not balanced, then it does not globally solve the average consensus-problem using Protocol (A). This assertion is consistent with the counterexample given in Figure 2.

Proposition 3. *Consider a network of integrators with directed information flow $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ that is strongly connected. Then, the digraph G globally asymptotically solves the average-consensus problem using Protocol* (A) *if and only if $\mathbf{1}^T L = 0$.*

Proof. From Theorem 1, with $w_r = \dfrac{1}{\sqrt{n}}\mathbf{1}$ we obtain

$$x^* = \lim_{t \to +\infty} x(t) = Rx_0 = w_r(w_l^T x_0) = \frac{1}{\sqrt{n}}(w_l^T x_0)\mathbf{1}.$$

This implies Protocol 1 globally exponentially solves a consensus problem with the decision value $\frac{1}{\sqrt{n}}(w_l^T x_0)$ for each node. If this decision value is equal to $\mathrm{Ave}(x_0), \forall x_0 \in \mathbb{R}^n$, then necessarily $\frac{1}{\sqrt{n}} w_l = \frac{1}{\sqrt{n}}$, i.e. $w_l = w_r = \frac{1}{\sqrt{n}}\mathbf{1}$. This implies that $\mathbf{1}$ is the left eigenvector of L. To prove the converse, assume that $\mathbf{1}^T L = 0$. Let us take $w_r = \frac{1}{\sqrt{n}}\mathbf{1}$, $w_l = \beta\mathbf{1}$ with $\beta \in \mathbb{R}, \beta \neq 0$. From condition $w_l^T w_r = 1$, we get $\beta = \frac{1}{\sqrt{n}}$ and $w_l = \frac{1}{\sqrt{n}}\mathbf{1}$. This means that the decision value for every node is $\frac{1}{\sqrt{n}}(w_l^T x_0) = \frac{1}{n}\mathbf{1}^T x_0 = Ave(x_0)$. □

The following result shows that if one of the agents uses a relatively small update rate (or step-size), i.e. $\gamma_{i^*} \gg \gamma_i$ for all $i \neq i*$. Then, the value of all nodes converges to the value of x_i^*. In other words, the agent i^* plays the role of a leader in *leader-follower* type architecture.

Corollary 1. *(multi-rate integrators) Consider a network of multi-rate integrator with the node dynamics*

$$\gamma_i \dot{x}_i = u_i, \quad \gamma_i > 0, \forall i \in \mathcal{I} \tag{11}$$

Assume each node applies Protocol (A). *Then, an agreement is globally asymptotically reached and the decision value of the group is*

$$\alpha = \frac{\sum_i \gamma_i x_i(0)}{\sum_i \gamma_i} \tag{12}$$

Proof. See [10]. □

Proposition 4. *Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a digraph with an adjacency matrix $\mathcal{A} = [a_{ij}]$ satisfying $a_{ii} = 0, \forall i$. Then, all the following statements are equivalent: i) G is balanced, ii) $\mathbf{1}^T L = 0$, and iii) $\sum_{i=1}^n u_i = 0, \forall x \in \mathbb{R}^n$ with $u = -Lx$.*

Proof. See [10]. □

Notice that in Proposition 4, the graph G does not need to be connected.

4.2 Performance and Mirror Graphs

In this section, we discuss performance issues of Protocol (A) with balanced graphs. An important consequence of Proposition 4 is that *for networks with balanced information flow, $\alpha = Ave(x)$ is an invariant quantity.* This is certainly not true for an arbitrary digraph. The invariance of $Ave(x)$ allows decomposition of x according to the following equation:

$$x = \alpha\mathbf{1} + \delta \tag{13}$$

where $\alpha = Ave(x)$ and $\delta \in \mathbb{R}^n$ satisfies $\sum_i \delta_i = 0$. We refer to δ as the (group) *disagreement vector.* The vector δ is orthogonal to $\mathbf{1}$ and belongs to an $(n-1)$-dimensional subspace called the *disagreement eigenspace* of L provided that G is strongly connected.

Moreover, δ evolves according to the (group) *disagreement dynamics* given by

$$\dot{\delta} = -L\delta. \tag{14}$$

It turns out that a useful property of balanced graphs is that for any balanced digraph G, there exists an undirected graph that has the same disagreement function as G. In the following, we formally define this induced undirected graph.

Definition 2. (mirror graph/operation) Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be weighted digraph. Let $\tilde{\mathcal{E}}$ be the set of *reverse edges* of G obtained by reversing the order of all the pairs in $\mathcal{E}$. The *mirror* of G denoted by $\hat{G} = \mathcal{M}(G)$ is an undirected graph in the form $\hat{G} = (\mathcal{V}, \hat{\mathcal{E}}, \hat{\mathcal{A}})$ with the same set of nodes as G, the set of edges $\hat{\mathcal{E}} = \mathcal{E} \cup \tilde{\mathcal{E}}$, and the symmetric adjacency matrix $\hat{\mathcal{A}} = [\hat{a}_{ij}]$ with elements

$$\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2} \geq 0 \tag{15}$$

The following result shows that the operations of $\mathcal{L}$ and *Sym* on a weighted adjacency matrix $\mathcal{A}$ commute. Moreover,

Theorem 3. *Let G be a digraph with adjacency matrix $\mathcal{A} = adj(G)$ and Laplacian $L = \mathcal{L}(G)$. Then $L_s = Sym(L) = (L + L^T)/2$ is a valid Laplacian matrix for $\hat{G} = \mathcal{M}(G)$ if and only if G is balanced, i.e. the following diagram commutes iff G is balanced*

$$\begin{array}{ccccc} G & \xrightarrow{adj} & \mathcal{A} & \xrightarrow{\mathcal{L}} & L \\ \mathcal{M}\downarrow & & Sym\downarrow & & Sym\downarrow \\ \hat{G} & \xrightarrow[adj]{} & \hat{\mathcal{A}} & \xrightarrow[\mathcal{L}]{} & \hat{L} \end{array} \tag{16}$$

Moreover, if G is balanced, the disagreement functions of G and $\hat{G}$ are equal.

Proof. We know that G is balanced iff $\mathbf{1}^T L = 0$. Since $L\mathbf{1} = 0$, we have $\mathbf{1}^T L = 0 \iff \frac{1}{2}(L + L^T)\mathbf{1} = 0$. Thus, G is balanced iff L_s has a right eigenvector of $\mathbf{1}$ associated with $\lambda = 0$, i.e. L_s is a valid Laplacian matrix. Now, we prove that $L_s = \mathcal{L}(\hat{G})$. For doing so, let us calculate $\hat{\Delta}$ element-wise, we get

$$\begin{aligned} \hat{\Delta}_{ii} &= \sum_j \frac{a_{ij} + a_{ji}}{2} = \frac{1}{2}(\deg_{out}(v_i) + \deg_{in}(v_i)) \\ &= \deg_{out}(v_i) = \Delta_{ii} \end{aligned}$$

Thus, $\hat{\Delta} = \Delta$. On the other hand, we have

$$L_s = \frac{1}{2}(L + L^T) = \Delta - \frac{A + A^T}{2} = \hat{\Delta} - \hat{A} = \hat{L} = \mathcal{L}(\hat{G})$$

The last part simply follows from the fact that $\hat{L}$ is equal to the symmetric part of L and $x^T(L - L^T)x \equiv 0$. ∎

Notation. For simplicity of notation, in the context of algebraic graph theory, $\lambda_k(G)$ is used to denote $\lambda_k(\mathcal{L}(G))$.

Now, we are ready to present our main result on performance of the Protocol (A) in terms of the speed of reaching a consensus as a group.

Theorem 4. *(performance of agreement) Consider a network of integrators with a directed information flow G that is balanced and strongly connected. Then, given Protocol* (A)*, the following statements hold: i) the group disagreement (vector) δ as the solution of the disagreement dynamics in* (14) *globally asymptotically vanishes with a speed that is equal to $\kappa = \lambda_2(\hat{G})$ (or the Fiedler eigenvalue of the mirror graph of G), i.e.*

$$\|\delta(t)\| \leq \|\delta(0)\| \exp(-\kappa t), \tag{17}$$

and ii) the following positive definite function

$$V(\delta) = \frac{1}{2}\|\delta\|^2 \tag{18}$$

is a valid Lyapunov function for the disagreement dynamics.

Proof. See [10]. ∎

A well-known observation regarding the Fiedler eigenvalue of an undirected graph is that for dense graphs λ_2 is relatively large and for sparse graphs λ_2 is relatively small [4] (this is why λ_2 is called the algebraic connectivity). According to this observation, from Theorem 4, one can conclude that a network with dense interconnections solves an agreement problem faster than a connected but sparse network.

4.3 Networks with Switching Topology

Consider a network of mobile agents that communicate with each other and need to agree upon a certain objective of interest or perform synchronization. Since, the nodes of the network are moving, it is not hard to imagine that some of the existing communication links can fail simply due to the existence of an obstacle between two agents. The opposite situation can arise where new links between nearby agents are created because the agents come to an effective range of detection with respect to each other. In other words, in the graph G representing the information flow of the network, certain edges can be added or removed from G. Here, we are interested to investigate that in case of a *network with switching topology* whether it is still possible to reach a consensus or not.

Consider a *hybrid system* with a continuous-state $x \in \mathbb{R}^n$ and a discrete-state G that belongs to a finite set of digraphs Γ_n. This Γ_n is the set of all digraphs of order n that are both *strongly connected* and *balanced*, i.e.

$$\Gamma_n = \{G = (\mathcal{V}, \mathcal{E}, \mathcal{A}) : \mathrm{rank}(\mathcal{L}(G)) = n-1, \mathbf{1}^T \mathcal{L}(G) = 0\}. \quad (19)$$

Given the node dynamics and the protocol, the continuous-state of the system evolves according to the following dynamics

$$\dot{x}(t) = -\mathcal{L}(G_k)x(t), \quad k = s(t), G_k \in \Gamma_n \quad (20)$$

where $s(t) : \mathbb{R}_{\geq 0} \to \mathcal{I}_{\Gamma_n}$ is a *switching signal* and $\mathcal{I}_{\Gamma_n} \subset \mathbb{N}$ is the index set associated with the elements of Γ_n. Clearly, Γ_n is a finite set, because either a digraph has no edges or it is a complete graph with $n(n-1)$ directed edges.

The key in solving the agreement problem for mobile networks with switching topology is a basic property of the Lyapunov function in (18) and the properties of balanced graphs. Note that the function $V(\delta) = \frac{1}{2}\|\delta\|^2$ does not depend on G or $L = \mathcal{L}(G)$. This property of $V(\delta)$ makes it an appropriate candidate as a *common Lyapunov function* for stability analysis of the switching system (20).

Theorem 5. *For any arbitrary switching signal $s(\cdot)$, the solution of the switching system (20), globally asymptotically converges to $Ave(x(0))$ (i.e. average-consensus is reached). Moreover, the following smooth, positive definite, and proper function*

$$V(\delta) = \frac{1}{2}\|\delta\|^2 \quad (21)$$

is a valid common Lyapunov function for the disagreement dynamics given by

$$\dot{\delta}(t) = -\mathcal{L}(G_k)\delta(t), \quad k = s(t), G_k \in \Gamma_n \quad (22)$$

Furthermore, the disagreement vector δ vanishes exponentially fast with the least rate of

$$\kappa^* = \min_{G \in \Gamma_n} \lambda_2(\mathcal{L}(\hat{G}))) \quad (23)$$

In other words, $\|\delta(t)\| \leq \|\delta(0)\| \exp(-\kappa^* t)$.

Proof. Due the fact that G_k is balanced for all k and $u = -\mathcal{L}(G_k)x$, we have $\mathbf{1}^T u = -(\mathbf{1}^T \mathcal{L}(G_k))x \equiv 0$. Thus, $\alpha = Ave(x)$ is an invariant quantity which allows us to decompose x as $x = \alpha\mathbf{1} + \delta$. Therefore, the disagreement switching system induced by (20) takes the form (22). Calculating $\dot{V}$, we get

$$\begin{aligned} \dot{V} &= -\delta^T \mathcal{L}(G_k)\delta = -\delta^T \mathcal{L}(\hat{G}_k)\delta \\ &\leq -\lambda_2(\mathcal{L}(\hat{G}_k))\|\delta\|^2 \leq -\kappa^*\|\delta\|^2 \\ &= -2\kappa^* V(\delta) < 0, \forall \delta \neq 0 \end{aligned} \quad (24)$$

This guarantees that $V(\delta)$ is a valid common Lyapunov function for the disagreement switching system (22). Moreover, we have

$$V(\delta(t)) \leq V(\delta(0)) \exp(-2\kappa^* t)$$

and the upper bound on $\|\delta(t)\|$ follows. Finally, the minimum in (23) always exists and is achievable because Γ_n is a finite set. □

5 Simulation Results

Figure 4 shows four different networks each with $n = 10$ nodes that are all strongly connected and balanced. The weights associated with all the edges are 1. For an initial node values satisfying $Ave(x(0)) = 0$, we have plotted the state trajectories and the disagreement function $\|\delta\|^2$ associated with these four digraphs in Figure 5. It is clear that as the number of the edges of the graph increase, algebraic connectivity (or λ_2) increases, and the settling time of the trajectory of the node values decreases. The case of a directed cycle of length $n = 10$, or G_a, has the largest over-shoot. In all four cases, an agreement is asymptotically reached and the performance is improved as a function of $\lambda_2(\hat{G}_k)$ for $k \in \{a, b, c, d\}$.

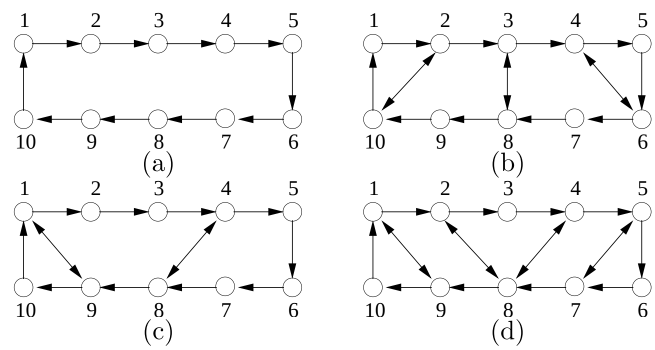
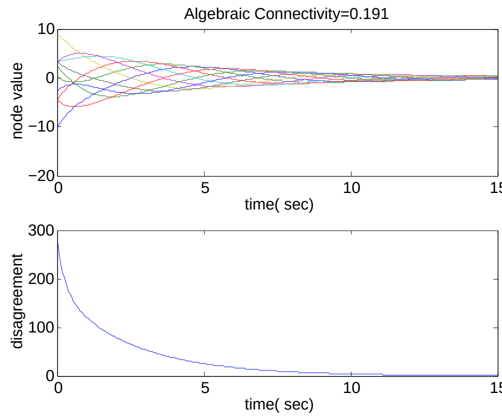


Figure 4: For examples of balanced and strongly connected digraphs: (a) G_a, (b) G_b, (c) G_c , and (d) G_d satisfying.

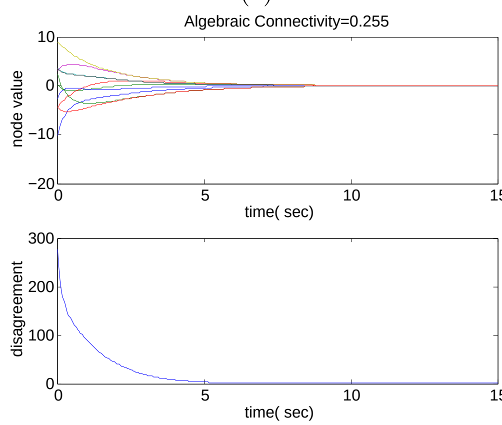
In Figure 6(a), a finite-state machine is shown with the set of states $\{G_a, G_b, G_c, G_d\}$ representing the discrete-states of a network with switching topology as a hybrid system. The hybrid system starts at the discrete-state G_b and switches every $T = 1$ second to the next state according to the state machine in Figure 6(a). The continuous-time state trajectories and the group disagreement (i.e. $\|\delta\|^2$) of the network are shown in Figure 6(b). Clearly, the group disagreement is monotonically decreasing. One can observe that an average-consensus is reached asymptotically. Moreover, the group disagreement vanishes exponentially fast.

6 Conclusion

In this paper, we addressed convergence and performance problems for an agreement protocol for a network of dynamic agents with integrator dynamics and directed information flow. Moreover, we analyzed robustness of this consensus protocol to changes in the topology of the network. We showed that balanced



(a)



(b)

Figure 5: For examples of balanced and strongly connected digraphs: (a) G_a and (b) G_d.

graphs are the only type of digraphs that solve the average-consensus problem with the aforementioned agreement protocol. We also proved that for any balanced digraph, there exists an undirected graph called the mirror graph. Fiedler eigenvalue of the mirror graph is used to quantify the speed of convergence of a linear agreement protocol on digraphs. Moreover, a common Lyapunov function that allowed convergence analysis for agreement in balanced networks with switching topology.

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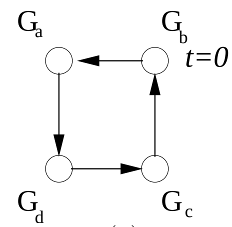
References

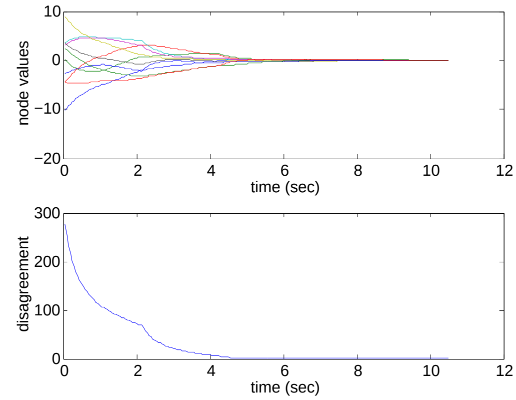
[1] J. P. Desai, J. P. Ostrowski, and V. Kumar. Modeling and control of formations of nonholonomic mobile robots. *IEEE Trans. on Robotics and Automation*, 17(6), December 2002.

[2] A. Fax and R. M. Murray. Graph laplacians and stabilization of vehicle formations. *The 15th IFAC World Congress*, June 2002.

[3] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.



(a)



(b)

Figure 6: (a) A finite-state machine with four states representing the discrete-states of a network and (b) trajectory of the node values and the group disagreement.

[4] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*. Springer, 2001.

[5] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1987.

[6] A. Jadbabaie, J. Lin, and S. A. Morse. Coordination of groups of mobile agents using nearest neighbor rules. *IEEE Trans. on Automatic Control*, 48(6):988–1001, June 2003.

[7] R. W. Lawton, J. R. T. Beard and B. J. Young. A Decentralized Approach to Formation Maneuvers. *IEEE Trans. on Robotics and Automation (to appear)*.

[8] N. A. Lynch. *Distributed Algorithms*. Morgan Kaufmann Publishers, Inc., 1997.

[9] M. Mesbahi and F. Y. Hadegh. Formation flying of multiple spacecraft via graphs, matrix inequalities, and switching. *AIAA Journal of Guidance, Control, and Dynamics*, 24(2):369–377, March 2000.

[10] R. Olfati Saber and R. M. Murray. Agreement problems in networks with directed graphs and switching topology. Technical Report CIT-CDS 03-005, California Institute of Technology, Control and Dynamical Systems, Pasadena, California, February 2003.

[11] R. Olfati Saber and R. M. Murray. Consensus Protocols for Networks of Dynamic Agents. *Proc. of the American Control Conference*, June 2003.