EUROTHERM 82
Numerical Heat Transfer 2005
September 13–16, 2005, Gliwice–Cracow, Poland
Eds.: A. Nowak, R.A. Białecki

EUROTHERM 82 – Modified Method of Characteristics for Solving the Transient Radiation Transfer Equation

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Key words: method of characteristics, transient radiation transfer, turbide media

Abstract
This paper presents the modified method of characteristics for simulating multidimensional transient radiation transfer in absorbing and scattering media. The modified method of characteristics transforms the integro-differential equation for radiative transfer expressed in terms of both space and time into four ordinary differential equations with respect to time. It makes use of an arbitrary set of points and, unlike the conventional method of characteristics follows the photons backward in space along their characteristics curves (or pathline). First, the principle and advantages of the numerical scheme are presented. Then, test problems for diffuse and collimated irradiation in one- and three-dimensional participating media with various boundary conditions are considered. The numerical results show good agreement with analytical and numerical solutions reported in literature. The scheme is fast and able to capture the sharp discontinuities associated with the propagation of a radiation front in transient radiation transport.

1 Introduction

Ultra-short pulsed lasers are used in a wide variety of applications such as thin film property measurements, laser assisted micro-machining, laser removal of contamination particles from surfaces, optical data storage, optical ablation and ablation of polymers [1]. Ultra-short pulsed lasers are also used in remote sensing of the atmosphere, combustion chambers and other environments which involve interaction of the laser beam with scattering and absorbing particles of different sizes. Another interesting application of short-pulsed lasers is in biomedical optical tomography where their use can potentially provide physiological and morphological information about the interior of living tissues and organs in a non-invasive manner.

All the applications described above require models to predict transient radiation transport in participating media. In the past, some analytical studies of transient radiative transfer have been conducted and reviewed by Mitra and Kumar [2]. They examined the transport of light pulses through absorbing-scattering media with different approximate mathematical models.

The governing equation for transient radiation transfer in an absorbing and scattering medium is the radiative transfer equation (RTE). The RTE expresses an energy balance in a unit solid angle of $d\Omega$ about the direction $\hat{\mathbf{s}}$ within a wavelength interval $\mathrm{d}\lambda$ about λ. It can be written as [3],

$$\frac{1}{c}\frac{\partial I_\lambda}{\partial t} + (\hat{\mathbf{s}} \cdot \nabla) I_\lambda = \kappa_\lambda I_{b\lambda} - \kappa_\lambda I_\lambda - \sigma_{s\lambda} I_\lambda + \frac{\sigma_{s\lambda}}{4\pi}\int_{4\pi} I_\lambda(\hat{\mathbf{s}}_i)\Phi_\lambda(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\mathrm{d}\Omega_i \quad (1)$$

where I_λ is the intensity in the $\hat{\mathbf{s}}$ direction and c, the speed of light in the medium. The linear absorption and scattering coefficients are denoted by κ_λ and $\sigma_{s\lambda}$, respectively. The scattering phase function $\Phi_\lambda(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})$ represents the probability that radiation propagating in the solid angle $\mathrm{d}\Omega_i$ direction around $\hat{\mathbf{s}}_i$ be scattered into the cone $\mathrm{d}\Omega$ around the direction $\hat{\mathbf{s}}$. The RTE is an integro-differential equation involving seven independent variables and radiation characteristics $\sigma_{s\lambda}$, κ_λ, and Φ_λ of materials which may depend on wavelength, temperature, and location. Thus, exact solutions of the RTE are difficult and exact analytical solutions exist for only a few simple cases [3].

The commonly used methods to solve the transient radiative transfer equation are the Monte Carlo method, the integral equation solution, the finite volume method (FVM), the radiation element method (REM) [4], and the discrete ordinates method (DOM).

The Monte Carlo method is often used to simulate problems involving radiative heat transfer because of its simplicity, the ease by which it can be applied to arbitrary configurations and its ability to capture actual and often complex physical conditions [5]. The Monte Carlo technique has been used by Guo *et al.* [5] to simulate short-pulsed laser transport in anisotropically scattering and absorbing media. The Monte Carlo method has also been widely used in biomedical optics to simulate steady state laser transport in biological tissue [6, 7]. However, the method has inherent statistical errors due to its stochastic nature [3]. It is also computationally time consuming and demands a lot of computer memory as the histories of the photons have to be stored at every instant of time [5].

The backward or reverse Monte Carlo has been developed as an alternative approach when solutions are needed only at particular locations and times [8, 9]. The method is similar to the traditional Monte Carlo method, except that the photons are tracked in a time-reversal manner. The method was successfully applied by Lu and Hsu [9] to simulate transient radiative transport in a non-emitting, absorbing, and anisotropically scattering one-dimensional slab subjected to ultra-short light pulse irradiation.

Analytical solutions of the radiation transfer equation in integral form for inhomogeneous and non-scattering medium have been obtained for 1-D [10] and 3-D geometries [11]. Then, Tan and Hsu [12] used the integral equation formulation to simulate radiative transport in 1-D absorbing and isotropically scattering media with black boundaries exposed to diffuse or collimated irradiation. The authors extended the method to solve the same problem in 3-D geometries [13]. Wu [14] used the integral equation to compute the temporal reflectivity and transmissivity of 1-D absorbing and isotropically scattering slabs with various scattering albedos and optical thicknesses which compared well with results obtained using the Monte Carlo method.

Chai and co-workers [15, 16] used the finite volume method to solve the transient RTE. They used the finite volume technique with the “step” and CLAM spatial discretization schemes to model transient radiative transfer in 1-D and 3-D geometries [15, 16]. The authors found that the CLAM scheme captures the penetration depths of radiation more accurately than the “step” scheme for the same grid.

Finally, the discrete ordinates method has been used by various researchers to solve the transient RTE. Sakami *et al.* [17] used the DOM to analyze the ultra-short light pulse propagation in a 2-D anisotropically scattering medium. Guo and Kumar [18] used it to simulate short-pulse laser transport in 2-D anisotropically scattering turbid media. They later extended the technique to solve for 3-D geometries and compared the results with Monte Carlo simulations [19]. They found that the transient discrete ordinates method cannot capture the abrupt changes in the transmittance as predicted by the Monte Carlo method. Guo and co-workers [20, 21] further used the DOM in 3-D geometries to model transport of ultrafast laser pulses and fluorescence in heterogeneous biological tissues for the purpose of detecting inhomogeneities in otherwise homogeneous tissue.

2 Modified Method of Characteristics

The conventional method of characteristics (or direct marching method) is commonly used to solve hyperbolic partial differential equations which often occur in compressible fluid flow. It is based on the Lagrangian formulation, which identifies photons at initial time $t = t_0$ and follows them along the characteristic at subsequent times as they are transported. Characteristics are pathlines of photons in physical space along which information propagates. Though the direct method results in accurate solutions, it has several disadvantages. Time increments along different characteristic curves may be different and so the solution may be obtained at different times on each characteristic curve. Also, the characteristic curves may coalesce or spread apart due to non-uniform velocities resulting in a highly distorted grid. The modified method of characteristics on the other hand, follows photons backward in space and uses any arbitrary pre-specified set of points. Thus, the solution is obtained at the same times at all grid points and overcomes the problems related to grid deformation (see Ref. [22] and references therein). Consider a Cartesian coordinate system, the characteristic curve in physical space is defined by,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c\sin\theta\cos\phi, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = c\sin\theta\sin\phi, \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = c\cos\theta \tag{2}$$

By definition, the total derivative of $I_\lambda(x, y, z, t)$ can be written as,

$$\frac{\mathrm{D}I_\lambda}{\mathrm{D}t} = \frac{\partial I_\lambda}{\partial t} + \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial I_\lambda}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial I_\lambda}{\partial y} + \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\partial I_\lambda}{\partial z} \tag{3}$$

Then, along the characteristic curves in (x, y, z, t) space, the RTE [Equation (1)] simplifies to,

$$\frac{1}{c}\frac{\mathrm{D}I_\lambda}{\mathrm{D}t} = -\kappa_\lambda I_\lambda - \sigma_{s\lambda} I_\lambda + \kappa_\lambda I_{b\lambda} + \frac{\sigma_{s\lambda}}{4\pi}\int_{4\pi} I_\lambda(\hat{\mathbf{s}}_i)\Phi_\lambda(\hat{\mathbf{s}}_i, \hat{\mathbf{s}})\mathrm{d}\Omega_i \tag{4}$$

Thus, the spatio-temporal partial integro-differential Equation (1) is converted into 3 ordinary differential equations in time, [Equation (2)] and 1 temporal integro-differential equation [Equation (4)]. Figure 1 shows a 3-D computational cell in Cartesian coordinates.

The modified method of characteristics consists of determining the coordinates (x_n, y_n, z_n) of the point in space from where the particles located at the grid point (x_a, y_b, z_c) at time $t + \Delta t$ originate from at time t while traveling in direction of polar angle θ_n and azimuthal angle ϕ_m. In other words, for each point of a specified grid, the pathline is projected rearward along the characteristic curve to the initial data surface to determine the initial data point. For example, in Figure 1, the point (x_a, y_b, z_c) is the point $(x_{i+1}, y_{j+1}, z_{k+1})$. The solid line represents the section of the characteristic curve along which the

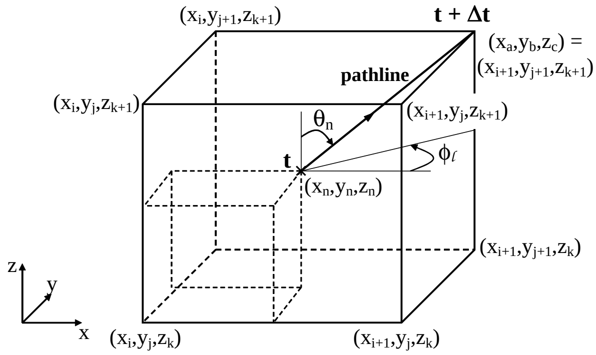

Figure 1: Typical computational cell used for inverse marching method containing the pathline of the photons.

photon traveled from location (x_n, y_n, z_n) to location (x_a, y_b, z_c) during the time interval between t and $t + \Delta t$.

To solve Equations (2) to (4), the radiation intensities and temperatures are initialized at all points in the computational domain. Then, for a given polar angle θ_n, an azimuthal angle ϕ_l, and for all internal grid points (x_a,y_b,z_c) where photons are present at time $t + \Delta t$, the position of the photon at time t is calculated as,

$$x_n = x_a - c \sin\theta_n \cos\phi_l \Delta t, \qquad y_n = y_b - c \sin\theta_n \sin\phi_l \Delta t, \qquad z_n = z_c - c\cos\theta_n \Delta t \tag{5}$$

The values of the variables I_λ at (x_n, y_n, z_n) and time t are obtained by Lagrangian interpolation using their values at time t at the eight corners of the computational cell in which the point (x_n, y_n, z_n) is located (Figure 1).

Then, Equation (4) is solved forward in time by the fourth order Runge-Kutta method at location $(x_a$, y_b, $z_c)$ and time $t + \Delta t$. The integral on the right hand side of Equation (4) is estimated by the 3/8 Simpson numerical integration. Finally, the boundary conditions are imposed in directions pointing toward the medium depending on whether the boundary is black, specularly or diffusely reflecting. For directions leaving the computational domain (outflow), the intensities at the boundary are computed just like any other internal point. The calculations are repeated for all the discretized values of polar and azimuthal angles.

The modified method of characteristics has the following main advantages: **(1)** Unlike finite volume techniques which propagate the information along the coordinate axis, the modified method of characteristics propagates information along the photon pathlines. Like the Monte Carlo method, it respects the physics of radiative transport resulting in accurate numerical results. **(2)** Since the method uses any arbitrary pre-specified set of points, it can be easily coupled with other numerical techniques such as finite volume, finite element or finite difference schemes. This is a valuable feature in situations involving multiple energy carriers and transport processes. **(3)** It does not require any outflow boundary conditions. The radiative transport equation is a hyperbolic equation and information propagates with finite speed, *i.e.*, the speed of light in the medium. In such equations, the solution at a point is determined only by the characteristics from upstream portion of the solution domain.

There are also a few disadvantages in using the modified method of characteristics to solve the RTE. The backward projected characteristic curves do not necessarily intersect the known solution surface at the pre-specified grid points and so the initial data at the backward projected characteristics must be determined by interpolation. This takes up computational effort and introduces interpolation errors into the solution.

The method described above could be considered as a hybrid method between the traditional discrete ordinates and the ray tracing methods. It is similar to the discrete ordinates method in that the RTE is solved along arbitrary directions. However, the modified method of characteristics converts the RTE into ordinary differential equations in time and solved along the characteristics, as opposed to the conventional implementation of the DOM [3], where the RTE in the form of a partial differential equation is solved along the grid lines. The present approach is comparable to that used by Coelho [23] to solve the RTE using the discrete ordinates method. The author determined the dependent variable I_λ by the values at points located at the intersection of the direction of propagation of radiation with the grid lines or surfaces, as opposed to directly using the grid nodes in the conventional DOM. The present method also differs from that used by Coelho in the sense that, the photons are traced back a distance which they would travel in one time step rather than all the way to the grid lines or surfaces.

Finally, unlike ray-tracing methods, the modified method of characteristics does not trace photon bundles from the source to the absorption point or to the boundaries. Instead photon bundles are traced backward in space only for the time interval between t and $t + \Delta t$. At a new time step, new photon bundles are traced back from all grid points and this procedure is repeated for all time steps.

3 Results and Discussion

For validation purposes, the numerical results obtained with the modified method of characteristics for a set of test cases have been compared with analytical solutions or numerical results reported in the literature using different numerical schemes. For the sake of clarity, spectral dependencies were not considered in these cases but could have been included without any modifications in the methodology. The spectral dependencies can be accounted for by using the modified method of characteristics at multiple wavelengths or in combination with band models [3]. The test cases considered consist of simulations of transient radiation transfer in absorbing and isotropically scattering cold media namely, (1) three-dimensional media exposed to diffuse irradiation, (2) a 1-D plane parallel slab irradiated by continuous collimated radiation, and (3) a 1-D plane parallel slab irradiated by pulsed collimated radiation. For 1-D problems, a discretization of N_z points along the $z-$direction and N_θ discrete directions for θ varying from 0 to π was used. In the case of 3-D problems, a discretization of $N_x \times N_y \times N_z$ along the x, y and $z-$directions respectively was used and the angular space of θ varying from 0 to π and ϕ varying from 0 to 2π was discretized into $N_\theta \times N_\phi$ directions.

3.1 3-D Transient Radiative Transfer in Scattering, Absorbing, and Emitting Media Exposed to Diffuse Irradiation

Let us consider the case of a cubic enclosure of a non-emitting, absorbing and isotropically scattering medium of thickness L. It is subjected to a transient unit step function emissive power on one side ($z = 0$) and the other side ($z = L$) being cold with optical thickness $\tau_L = 1$ defined as $\tau_L = \beta L$ where $\beta = \sigma_s + \kappa$ and the scattering albedo $\omega = \sigma_s/\beta = 0.1$. Initially, the medium is assumed to be at 0K and initial intensities everywhere in the medium are zero. Then, at t=0, the intensity at the wall, $z = 0.0$ in all directions pointing into the medium is set to 1.0 W/m^2.sr. The remaining walls are black and cold ($T = 0K$).

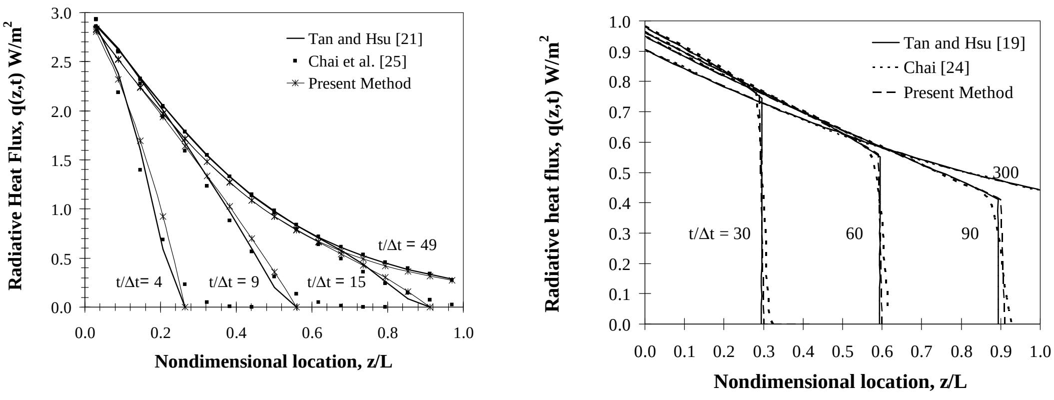

Figure 2: Radiative heat flux distribution at different times (left) in a homogeneous 3-D medium with a diffusely emitting boundary for $\tau = 1.0$ and $\omega = 0.1$, and (right) in a 1-D homogeneous medium exposed to continuous collimated irradiation.

A grid size of $35 \times 35 \times 35$ points along the x, y and $z-$directions was used. The entire angular space of θ varying from 0 to π and ϕ varying from 0 to 2π was divided into 30×24 discrete directions. Tan and Hsu [13] solved the same problem using the integral solution. The authors verified the reliability of the integration scheme, namely the DRV method by comparing their results with those obtained by using the YIX method. They used a grid size of $17 \times 17 \times 17$ volume elements in the x, y and $z-$directions respectively in all the cases and 1982 angular quadrature points for the YIX method. Similarly, Chai *et al.* [16] solved this problem using the finite volume method. Figure 2 compares the heat flux obtained using the present method along the center of the cube with those obtained by (i) Tan and Hsu [13] with the integral solution using the DRV scheme and (ii) Chai *et al.* [16] using a grid size of $17 \times 17 \times 17$, an angular discretization of 16×12 and making use of the CLAM scheme. In order to quantify the relative difference in the numerical results, we assumed that the values reported by Tan and Hsu [13] are converged and correspond to the exact solution. For steady-state, the relative error for the results reported by Chai *et al.* [16] compared to Tan and Hsu [13] was less than 2.5% while they were less than 6% for the present method. For the intermediate transients, the results reported by Chai *et al.* [16] result in infinite errors beyond the wavefront. Chai *et al.* [16] found that the finite volume method suffers from false scattering and cannot capture the wavefront accurately. In contrast, the present method is able to accurately capture the wavefront. However, there were large errors of up to 83% at the point right before the wavefront where the heat flux is small. It was less than 6% at all other points. This could be attributed to the fact that the solution was not converged in terms of the grid size or the number of directions. In order to minimize the error, a further refinement in the grid size was attempted, but was out of bounds in terms of memory and processing power for the single processor computer used. Refinement in the grid size and directions is anticipated to reduce the error.

3.2 1-D Transient Radiative Transfer With Collimated Irradiation

To solve the radiative transport equation for collimated irradiation, the intensity is split into two parts, (i) the radiation scattered away from the collimated radiation and (ii) the remaining collimated beam after partial extinction by absorption and scattering along its path. The contribution from emission is usually negligible compared to the incident and scattered intensity. Thus, the intensity for a gray medium is written as, $I(\mathbf{r}, \hat{\mathbf{s}}, t) = I_c(\mathbf{r}, \hat{\mathbf{s}}, t) + I_d(\mathbf{r}, \hat{\mathbf{s}}, t)$ [3]. Consider a plane parallel slab of an absorbing and isotropically scattering medium with constant and uniform optical properties exposed to time-dependent collimated radiation. In this case, The collimated intensity I_c, remnant of the incident irradiation $I_i(\mathbf{r}_w, t)$ is given by [24], $I_c(z, \hat{\mathbf{s}}, t) = I_i(t - z/c)\delta[\hat{\mathbf{s}} - \hat{\mathbf{s}}_c]e^{-\beta z}$ where $\hat{\mathbf{s}}_c$ corresponds to $\cos\theta = 1$. In addition, the governing equation for the non-collimated radiation intensity I_d can be written as,

$$\frac{1}{c}\frac{\mathrm{D}I_d(z, \hat{\mathbf{s}}, t)}{\mathrm{D}t} = -\beta I_d(z, \hat{\mathbf{s}}, t) + \frac{\sigma_s}{4\pi}\int_{4\pi} I_d(z, \hat{\mathbf{s}}_i, t)\mathrm{d}\Omega_i + \frac{\sigma_s}{4\pi} I_i(t - z/c)e^{-\beta z}H(t - z/c) \qquad (6)$$

where H is the Heaviside step function [$H(u) = 0$ if $u < 0$ and $H(u) = 1$ if $u \geq 0$]. while $\hat{\mathbf{s}}$ depends only on the polar angle θ, and the boundary conditions are $I_d(z = 0, \cos\theta > 0, t) = 0$ and $I_d(z = L, \cos\theta < 0, t) = 0$. The heat flux is computed using

$$\mathbf{q} = \int_{4\pi} I_d(\hat{\mathbf{s}})\hat{\mathbf{s}}\mathrm{d}\Omega + I_i(\mathbf{r}_w, t - s/c)\exp\left[-\int_0^s \beta(\mathbf{r} - s'\hat{\mathbf{s}}_c)ds'\right] H(t - s/c)\hat{\mathbf{s}}_c \qquad (7)$$

Two different incident radiations were considered for comparison with solutions reported in literature.

Continuous Collimated Pulse. The first case is a continuous collimated pulse corresponding to $I_i(t) = 0$ W/m^2/sr, for $t < 0$ and $I_i(t) = 1$ W/m^2/sr for $t \geq 0$. A converged solution was reached for a spatial discretization of $N_z = 101$ points and an angular discretization of $N_\theta = 25$ directions. The time interval Δt is equal to $\Delta z/c$ where $\Delta z = L/100$. The CPU time taken was about 6 seconds on a Pentium 4, 2.80 GHz machine for 360 time steps. Figure 2 compares the radiative heat fluxes obtained with the present technique with those obtained by (i) Tan and Hsu [12] using the integral solution and by (ii) Chai [15], using the finite volume technique with the CLAM scheme and the present technique. As can be seen, the modified method of characteristics captures the sharp discontinuities better than the finite volume technique. It should also be noted that Chai [15] used 300 control volumes and 20×1 angles per quadrant equivalent to $N_\theta = 40$ angles for θ varying from 0 to π compared to 101 nodes and $N_\theta = 25$ in the present study.

Ultra-Short Collimated Pulse. The second incident radiation profile is a truncated Gaussian distribution with a peak intensity at $t = t_c$ and pulse width t_p expressed as $I_i(t) = I_0 \exp\left[-4\ln 2(t - t_c)^2/t_p^2\right] \quad \text{for } 0 < t < 2t_c \quad \text{and} \quad I_i(t) = 0 \quad \text{for } t \geq 2t_c$ Numerical convergence was achieved with a discretization of $N_z = 101$ and $N_\theta = 25$ for the case of $\tau_L = 0.5$ and $N_z = 201$ and $N_\theta = 25$ for the case of $\tau_L = 5.0$. The time interval Δt had little effect on the numerical results as long as $\Delta t \leq \Delta z/c$. Thus, it was set equal to $\Delta z/c$ where $\Delta z = L/(N_z - 1)$ and N_z is the number of gridpoints in the $z-$direction. After solving for the intensities in all directions at every grid point, the hemispherical reflectance $R(t)$ and transmittance $T(t)$ are computed using the following formulae,

$$R(t) = -2\pi\int_{-1}^{0} I_d(0, \mu, t)\mu\mathrm{d}\mu/I_0 \qquad (8)$$

$$T(t) = [2\pi\int_0^1 I_d(L, \mu, t)\mu\mathrm{d}\mu + I_i(t - L/c)e^{-\beta L}H(t - L/c)]/I_0 \qquad (9)$$

The integrals in the formulae for hemispherical reflectance and transmittance are computed using the 3/8 Simpson numerical rule. The CPU time taken for computing the transmittance for the case of $\tau_L = 5.0$ and $\omega = 1.0$ using a spatial discretization of $N_z = 201$ points and an angular discretization of $N_\theta = 25$ was about 41 seconds for a total dimensionless time $t^* = 40$ defined by $t^* = \beta ct$. The CPU time taken for computing the reflectance for the case of $\tau_L = 0.5$ and $\omega = 0.95$ using a spatial discretization of

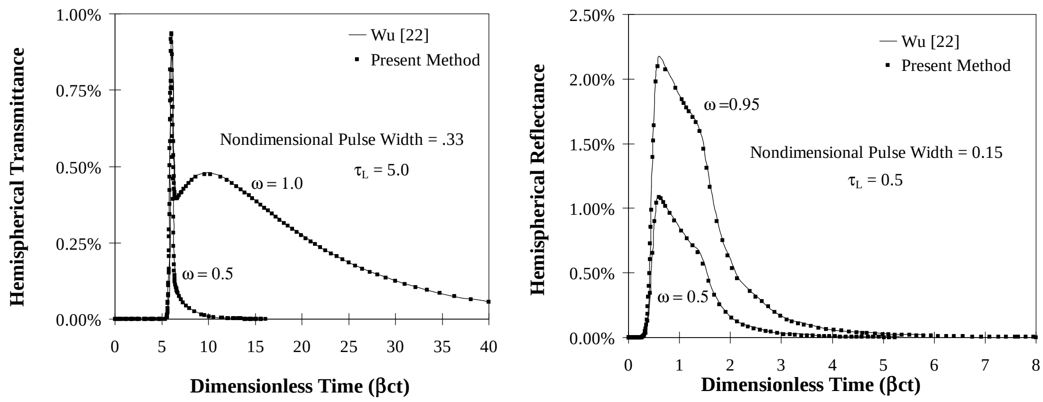

Figure 3: Time-resolved hemispherical transmittance (left) and reflectance (right) for $\tau_L = 5.0$, $t_c/t_p = 3$ and $\beta c t_p = 0.33$.

$N_z = 101$ points and an angular discretization of $N_\theta = 25$ directions per octant was about 21 seconds for a total dimensionless time of $t^* = 8$.

Figures 4 compares the results of the numerical integration for the transmittance and reflectance of homogeneous absorbing and isotropically scattering slabs obtained by Wu [14] from the analytical solution of the integral equation with those obtained with the modified method of characteristics. The plot on the left corresponds to the temporal transmittance of a slab of optical thickness $\tau_L = 5.0$ and scattering albedos $\omega = 1.0$ and $\omega = 0.5$. That on the right corresponds to the reflectance from a slab of optical thickness $\tau = 0.5$ and scattering albedos of 0.95 and 0.5. Good agreement is observed for both the transmittance and reflectance, and the mean error was less than 5% in all cases. Thus the modified method of characteristics can be used to simulate transport of collimated radiation in a fast and accurate manner.

3.3 Discussion

The results presented confirm the validity of the numerical scheme and its capability in handling various transient problems. It has been shown that the modified method of characteristics is a fast and accurate technique to simulate transient radiative transfer in absorbing and scattering media. It can also be easily modified to handle various other geometries and phase functions, thus enabling it to simulate radiative transfer in more complex situations such as those for biomedical applications. The computer program used to implement the described method has not been optimized neither has a thorough error and stability analysis been done. Instead, the study has been aimed at demonstrating the applicability of the method to a range of problems encountered in transient radiation transfer. A careful study of the errors introduced due to the Lagrangian interpolation should be performed. This would be helpful in comparing the accuracy of this method to other methods like the finite volume method which is prone to false scattering [16]. Also, various improvements can be made to decrease the computational time: **(1)** The number of discrete directions can be reduced or replaced by quadrature as commonly used in the discrete ordinate method to accelerate the computation of the in-scattering term. **(2)** To further accelerate the computations, the radiation wavefront can be tracked and computations be performed only at points through which the wave has passed. **(3)** Though the case of a diffusely reflecting boundary condition was not discussed here, it can be easily implemented as done by Rukolaine *et al.* [25]. Also, specularly reflecting boundaries using the modified method of characteristics have been successfully implemented recently for phonon transport [26]. **(4)** Also, since the method is fully explicit it can be easily adapted for parallel computing. This could be a useful feature in real time analysis and inverse problems.

4 Concluding Remarks

The modified method of characteristics has been presented as a scheme for solving the radiative transport equation. It has been shown that the method can handle various problems including multidimensional, transient radiative transport in media exposed to both collimated or diffuse irradiation. The method is fast and accurate and compares well with those obtained using other methods and reported in literature. In particular, the method was able to capture the sharp spatial discontinuities associated with transient radiative transport. Also, since the method makes use of any arbitrary fixed grid, it can be coupled easily with other methods to simulate multi-carrier energy transport or combined transport phenomena.

Acknowledgements

The authors are thankful to Professor Pei-feng Hsu and Professor John Chai for providing data for the integral equation solution and the finite volume solution.

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