Homework assignment #3

1. In each subproblem, give a formula or simple algorithm for evaluating the prox-operator

\[ \text{prox}_f(x) = \arg\min_u \left( f(u) + \frac{1}{2} \| u - x \|_2^2 \right). \]

(a) \( f(x) = \| x \|_1 \) with domain \( \text{dom} f = \{ x \mid \| x \|_\infty \leq 1 \} \).
(b) \( f(x) = \max_k x_k \).
(c) \( f(x) = \| Ax - b \|_1 \) where \( AA^T = D \) with \( D \) positive diagonal.

2. Find the prox-operator of the function \( f(X) = -\log \det X \) where \( X \in S^n \) and \( \text{dom} f = S^n_{++} \). Here, the prox-operator is defined as

\[ \text{prox}_f(X) = \arg\min_U \left( f(U) + \frac{1}{2} \| U - X \|_F^2 \right) \]

where \( \| \cdot \|_F \) is the Frobenius norm.

3. The Moreau envelope of a closed convex function \( f \) is defined as

\[ f_{(\lambda)}(x) = \inf_u \left( f(u) + \frac{1}{2\lambda} \| u - x \|_2^2 \right) \]

(lecture 10, page 14). Prove the following formula for the prox-operator of \( f_{(\lambda)} \):

\[ \text{prox}_{\mu f_{(\lambda)}}(x) = \frac{\lambda}{\lambda + \mu} x + \frac{\mu}{\lambda + \mu} \text{prox}_{(\lambda+\mu)f}(x). \]

As an example, applying this to \( f(x) = \| x \|_1 \) gives a formula for the prox-operator of the Huber penalty.