L. Vandenberghe

Homework assignment #3

1. In each subproblem, give a formula or simple algorithm for evaluating the proximal mapping
$$\mathrm{prox}_f(x) = \operatorname*{argmin}_u \left(f(u) + \frac{1}{2}\|u - x\|_2^2 \right).$$
   (a) $f(x) = \|x\|_1$ with domain $\mathbf{dom}\, f = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$.
   (b) $f(x) = \|Ax - b\|_1$ where $AA^T = D$ with D positive diagonal.
   (c) $f(x) = \max_k x_k$.
   (d) $f(x) = \|x\|_2$ with domain $\mathbf{R}^n_+$.

   Hints. For the function (a) the minimization in the definition of prox_f is separable. In problem (b), combine the property on page 8-8 with the scaling rule on page 8-4. The functions in (c) and (d) can be expressed as support functions.

2. Give the proximal mapping of the following two functions.
   (a) $f(X) = -\log\det X$ where $X \in \mathbf{S}^n$ and $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.
   (b) $f(X) = \|X\|_*$ where $X \in \mathbf{R}^{m\times n}$ and $\|\cdot\|_*$ is the trace norm (sum of singular values). This is the dual norm of the spectral norm (maximum singular value).

   We use the Frobenius norm $\|\cdot\|_F$ to define the proximal mappings of functions of matrices:
$$\mathrm{prox}_f(X) = \operatorname*{argmin}_U \left(f(U) + \frac{1}{2}\|U - X\|_F^2 \right).$$

3. The Moreau envelope of a closed convex function f is defined as
$$f_{(\lambda)}(x) = \inf_u \left(f(u) + \frac{1}{2\lambda}\|u - x\|_2^2 \right)$$
   (lecture 10, page 11). Prove the following formula for the proximal mapping of $f_{(\lambda)}$:
$$\mathrm{prox}_{\mu f_{(\lambda)}}(x) = \frac{\lambda}{\lambda+\mu} x + \frac{\mu}{\lambda+\mu} \mathrm{prox}_{(\lambda+\mu)f}(x).$$
   As an example, applying this to $f(x) = \|x\|_1$ gives a formula for the proximal mapping of the Huber penalty.