Homework assignment #4

1. Give a formula or simple algorithm for evaluating the proximal mapping

$$\mathrm{prox}_f(x) = \operatorname*{argmin}_u \left(f(u) + \frac{1}{2}\|u - x\|_2^2 \right)$$

of each of the following functions on $\mathbf{R}^n$.

   (a) $f(x) = \|x\|_1$ with domain $\mathbf{dom}\, f = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$.

   (b) $f(x) = \|Ax - b\|_1$ where $AA^T = D$ with D positive diagonal.

   (c) $f(x) = \max_{k=1,\ldots,n} x_k$.

   (d) $f(x) = \|x\|_2$ with domain $\mathbf{R}^n_+$.

   (e) $f(x) = \|Ax\|_2$, with A nonsingular.

Hints. For the function (a) the minimization in the definition of prox_f is separable. In problem (b), combine the property on page 6.8 with the scaling rule on page 6.4. The functions in (c) and (d) can be expressed as support functions, and the proximal operators follow from the property on page 6.18. The function in (e) is a norm and the proximal mapping can be computed via projection on the unit ball for the dual norm (page 6.19).

2. Give the proximal mapping of the following two functions.

   (a) $f(X) = -\log\det X$ where $X \in \mathbf{S}^n$ and $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

   (b) $f(X) = \|X\|_*$ where $X \in \mathbf{R}^{m\times n}$ and $\|\cdot\|_*$ is the trace norm (sum of singular values). This is the dual norm of the spectral norm (maximum singular value).

We use the Frobenius norm $\|\cdot\|_F$ to define the proximal mappings of functions of matrices:

$$\mathrm{prox}_f(X) = \operatorname*{argmin}_U \left(f(U) + \frac{1}{2}\|U - X\|_F^2 \right).$$