Homework assignment #6

1. In the lecture we derived ADMM from the Douglas–Rachford splitting method applied to a dual problem. One can also derive the Douglas–Rachford splitting from ADMM. Show that ADMM applied to the problem

$$\begin{array}{ll} \text{minimize} & f(x) + g(u) \\ \text{subject to} & x - u = 0 \end{array}$$

(with variables x and u) gives the Douglas–Rachford splitting method in its equivalent form on page 11.5.

2. Describe an efficient implementation of ADMM for each of the following four optimization problems with variable $x \in \mathbf{R}^n$.

We use the notation $H(x)$ for the linear function that maps an n-vector x to the $p \times q$ Hankel matrix

$$H(x) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_q \\ x_2 & x_3 & x_4 & \cdots & x_{q+1} \\ x_3 & x_4 & x_5 & \cdots & x_{q+2} \\ \vdots & \vdots & \vdots & & \vdots \\ x_p & x_{p+1} & x_{p+2} & \cdots & x_n \end{bmatrix},$$

for some fixed p, q with $p+q-1 = n$. The norm $\|H(x)\|_*$ is the trace norm (or nuclear norm) of the $p \times q$ matrix $H(x)$, *i.e.*, the sum of the singular values (see homework 4). The matrix D in parts (b) and (d) is the $(n-1) \times n$ finite-difference matrix

$$D = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}.$$

By "efficient implementation" we mean that the cost per iteration should be dominated by the cost of one singular value decomposition of a $p \times q$ matrix (assuming p and q are not small).

(a) Given $a \in \mathbf{R}^n$, solve

$$\text{minimize} \quad \|H(x)\|_* + \frac{1}{2}\|x - a\|_2^2.$$

(b) Given $a \in \mathbf{R}^n$ and $\gamma > 0$, solve

$$
\begin{array}{ll}
\text{minimize} & \|H(x)\|_* + \frac{1}{2}\|x - a\|_2^2 \\
\text{subject to} & \|Dx\|_2 \leq \gamma.
\end{array}
$$

(c) Given $a \in \mathbf{R}^n$, solve

$$
\text{minimize} \quad \|H(x)\|_* + \|x - a\|_1.
$$

(d) Given $a \in \mathbf{R}^n$ and $\gamma > 0$, solve

$$
\begin{array}{ll}
\text{minimize} & \|H(x)\|_* + \|x - a\|_1 \\
\text{subject to} & \|Dx\|_2 \leq \gamma.
\end{array}
$$