Exercises for week 9 discussion

The following exercises will be the topic of the discussion section.

1. *Perturbation lemma.* With the notation of page 16.3, show that if A is invertible and $\|A^{-1}B\| < 1$, then

$$\|(A+B)^{-1} - A^{-1}\| \leq \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|} \|A^{-1}\|.$$

2. [Deuflhard] Consider a nonlinear equation $f(x) = 0$ where $f : \mathbf{R}^n \to \mathbf{R}^n$ is differentiable. Suppose $f(\hat{x}) \neq 0$ and the Jacobian matrix $f'(\hat{x})$ of f at $\hat{x}$ is nonsingular. Show that the Newton direction $v = -f'(\hat{x})^{-1} f(\hat{x})$ is a descent direction of the function $g_A(x) = \|Af(x)\|_2^2$, for *any* nonsingular matrix A. In other words, show that

$$\nabla g_A(\hat{x})^T v < 0 \quad \text{for all nonsingular } A. \tag{1}$$

   Are there other directions v (other than the Newton direction) with this property?

3. *Local convergence of Newton's method.* The Kantorovich theorem in lecture 16 is called a *semi-local* convergence result, because it does not start from an assumption that a solution exists. If we make this assumption, local convergence is easier to prove.

   Let $x^\star$ be a solution at which $f'(x^\star)$ is invertible and define $\alpha = \|f'(x^\star)^{-1}\|$. Assume that f is differentiable in a neighborhood $B = \{x \mid \|x - x_\star\| < \rho\}$ and that f' is β-Lipschitz continuous on this set:

$$\|f'(x) - f'(y)\| \leq \beta \|x - y\| \quad \text{for all } x, y \in B.$$

   (a) Suppose $x_k \in B$ and $\|x_k - x^\star\| \leq 1/(2\alpha\beta)$. Use the perturbation result on page 16.3 to show that $f'(x_k)$ is invertible and that

$$\|f'(x_k)^{-1}\| \leq 2\alpha.$$

   (b) It can be shown that the Lipschitz continuity assumption implies that

$$\|f(y) - f(x) - f'(x)(y - x)\| \leq \frac{\beta}{2} \|y - x\|^2 \quad \text{for all } x, y \in B.$$

   Use this fact to prove that

$$\|x_{k+1} - x^\star\| \leq \alpha\beta \|x_k - x^\star\|^2.$$

   (c) Conclude that if $x_0 \in B$ and $\|x_0 - x^\star\| \leq 1/(2\alpha\beta)$, then the iterates x_k in Newton's method are well defined and converge to $x^\star$.